Zero Lower Bound Government Spending Multipliers and Equilibrium Selection

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Abstract

In the standard new Keynesian model the government spending multiplier under constant, zero nominal interest rates can be large and positive, or large and negative. Small changes in fiscal policy discontinuously switch the multiplier from one case to the other. This paper shows that this discontinuity occurs because the standard minimum state variable solution switches between two equilibrium selection rules. Using a consistent equilibrium selection rule, government spending multipliers vary continuously with the fiscal experiment. Thus, government spending multipliers may be much less sensitive to the design of fiscal policy than implied by existing work.

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1 Introduction

In the standard new Keynesian model the government spending multiplier under constant, zero nominal interest rates can be large and positive, or large and negative, depending on the fiscal shock. Small changes in the design of fiscal policy can switch the multiplier from one case to the other (Mertens and Ravn, 2014; Boneva, Braun, and Waki, 2016; Wieland, 2018).[1] Figure 1 provides an example: Increasing the persistence of the government spending shock from 0.47 to 0.53 causes a decline in the government spending multiplier from $+6.0$ to -6.0. The sharp discontinuity in government spending multipliers under constant, zero nominal interest rates is concerning for the use of new Keynesian framework in policy evaluation. Is it possible that a slight change in the design of fiscal programs could result in such a dramatic shift in their efficacy?

In this paper I show that the discontinuity in figure 1 is a consequence of implicitly switching between different equilibria. The figure plots one equilibrium type when the persistence of government spending is less than 0.504 and another equilibrium type when the persistence exceeds 0.504. When a consistent equilibrium selection criterion is used, then there is no discontinuity.

I use the continuous time new Keynesian model in Werning (2012) and Cochrane (2017) to characterize the government spending multiplier under constant nominal interest rates. The zero lower bound (or any non-zero lower bound) is one reason for why monetary policy may be passive, but my results apply more broadly to any circumstance in which nominal interest rates are constant.

I allow for a range of shock processes and different equilibrium selection rules. An advantage of the continuous time framework is that one can transparently separate the role of equilibrium selection from the role of the shock process in determining the government spending multiplier.

[1] In their comparative statics, Mertens and Ravn (2014) and Boneva et al. (2016) simultaneously change the persistence of fiscal policy and the persistence of the zero lower bound. As I show in Wieland (2018), it is the change in the fiscal experiment that accounts for the large changes in government spending multipliers.

I show that government spending multipliers under constant nominal interest rates vary continuously with the persistence of the fiscal experiment when equilibria are selected using the standard criterion (the Fed attains its zero inflation target upon exit) or the Cochrane (2017) backward-stable selection criterion (equilibria remain bounded as the duration of constant nominal interest rates becomes infinite). Under the standard equilibrium selection criterion, the government spending multiplier is always above 1 and finite if the duration of government spending is also finite as already shown in Cochrane (2017). Further, the multiplier is continuously differentiable and monotonic in the duration and persistence of the government spending shock. As the duration of the fiscal experiment approaches infinity (a diffusion process), the government spending multiplier may either approach an asymptote or diverge to infinity depending on parameters. In either case, the multiplier remains continuous in the limit.[2]

The government spending multiplier under the Cochrane (2017) backward-stable selection criterion is instead always below 1, but inherits similar continuity properties. It is finite if the duration of government spending is also finite, and it is continuously differentiable in both the duration and persistence of the fiscal shock. As the duration of the fiscal shock approaches infinity, the multiplier either asymptotes or approaches minus infinity depending on parameters. The multiplier again remains continuous in the limit. Thus, either the standard or the backward-stable selection criterion delivers continuous behavior of the government spending multiplier.

However, for autoregressive processes in linear models or jump processes, the literature typically solves for the government spending multiplier using the minimum state variable criterion. This criterion selects bounded, time-invariant solutions, such as those in in figure 1. I show that this criterion switches from a limit of the standard equilibrium to a limit of the backward-stable equilibrium at the point of discontinuity in figure 1. Thus, rather than being a fundamental property of the model, the discontinuity occurs because the figure is

[2] I use the order topology over the extended real line the extend the notion of continuity to $\pm\infty$.

plotting two different equilibria.

The minimum state variable criterion cannot consistently select one equilibrium type, because it only selects bounded solutions. But in figure 1 the corresponding limit of the standard equilibrium is unbounded to the right of the discontinuity, and the corresponding limit of the backward-stable equilibrium is unbounded to the left of the discontinuity.

One solution to avoid inconsistencies and discontinuities is to restrict the duration of shocks to be finite (but arbitrarily long). Then there exists a locally-unique bounded equilibrium, with continuous multipliers, under either the standard or the backward-stable selection criterion. The solution would no longer be independent of time, but it could be easily computed using either the methods in Bodenstein, Erceg, and Guerrieri (2017) and Coibion, Gorodnichenko, and Wieland (2012) (in discrete time) or Cochrane (2017) (in continuous time). Alternatively, one could keep using the minimum state variable criterion, at the cost of discarding the parameter space to either the left or the right of the discontinuity (Woodford, 2011; Eggertsson and Singh, 2016). A complication with this strategy is that the point of discontinuity varies with model parameters and may need to be found numerically.

While I focus on government spending shocks, my results apply more broadly to all exogenous disturbances in the standard new Keynesian model under constant nominal interest rates. For example, Mertens and Ravn (2014) and Boneva et al. (2016) highlight that a discontinuity similar to figure 1 exists when the minimum state variable criterion is applied to tax policy.

I do not make a case for choosing an equilibrium selection criterion. Christiano, Eichenbaum, and Johannsen (2016) advocate for standard selection criterion, and Cochrane (2017) advocates for the backward-stable equilibrium. The continuity and smoothness results hold conditional on either selection criterion. However, I do not know of work advocating a switch from one selection criterion to another at a particular point in the parameter space. At a minimum this suggests caution in the use of the minimum state variable criterion for equilibrium selection. More generally, and as emphasized by Cochrane (2017) and Christiano et al. (2016), equilibrium selection is an important determinant of government spending multipliers under constant nominal interest rates.

2 Model

The model is a standard new Keynesian model in continuous time (Werning, 2012; Cochrane, 2017). Since the model is standard, I only report the linearized first order condition in the text. Appendix A derives these conditions.

Throughout my analysis there is perfect foresight except for an unanticipated shock at time $t = 0$.

2.1 Structural equations Optimal consumption behavior is characterized by an Euler equation,

$$\frac{d}{dt}c_t = (i_t - \pi_t - \rho)$$

where c_t is the log deviation of consumption from steady-state, i_t is the net nominal interest rate, π_t is the net inflation rate, and ρ is the discount rate. Given log utility, consumption growth is exactly equal to the real interest rate net of the discount rate. Solved forward, and assuming a return to steady-state, consumption today is determined by the expected future path of real interest rates net of the discount rate,

$$c_t = \int_0^{\infty} (i_{t+s} - \pi_{t+s} - \rho) ds.$$

In other words, consumption is determined by intertemporal substitution.

Firms face quadratic price adjustment cost following Rotemberg (1982). Their optimal pricing behavior yields a new Keynesian Phillips curve,

$$\frac{d}{dt}\pi_t = \rho\pi_t - \kappa^* \left\{c_t + \xi_g s_g g_t\right\}$$

where g_t is the log deviation of government spending from steady-state. The term $c_t + \xi_g s_g g_t$

are the marginal cost of production. Solved forward, inflation is the discounted sum of current and future marginal costs,

$$\pi_t = \kappa \int_0^\infty e^{-\rho s} \left\{ c_{t+s} + \xi_g s_g g_{t+s} \right\} ds$$

The non-negative parameters κ^* and ξ_g are composites of the structural parameters. Exact expressions for a simple model are in appendix A. Variations in the set-up will change the mapping from structural parameters to κ^* and ξ_g, but my derivations remain accurate for given values of κ^* and ξ_g. The parameter s_g is the steady-state government spending share in output. I treat it as separate from ξ_g to simplify formulas for the government spending multiplier.

The resource constraint of the economy is,

$$y_t = s_g g_t + (1 - s_g) c_t$$

where y_t is the log deviation of output from steady-state.

Government spending is financed with lump-sum taxes, and the government budget is balanced at all times.

2.2 Disturbances At $t = 0$ the government spending process unexpectedly takes on a positive value $g_0 > 0$. Subsequently the government spending process is deterministic. It decays at rate $\theta_g \geq 0$ up until time $t = T_g$. At $t = T_g$ government spending jumps to its steady state value,

$$g_t = \begin{cases} e^{-\theta_g t} g_0 & \text{if } 0 \leq t < T_g \\ 0 & \text{if } t \geq T_g \end{cases} \tag{1}$$

Budget balance implies that the government spending shock is tax-financed.

2.3 Zero Lower Bound I assume that the nominal interest rate is at steady-state $i_t = \rho$ and unresponsive up to period T. This simplifies the algebra relative to a more complex scenario where a natural-rate shock creates a recession that pushes the economy to the zero lower bound (or any non-zero bound) for T periods. These two scenarios yield identical formulas for government spending multipliers because the model is linear conditional on constant nominal interest rates (Wieland, 2018). Intuitively, in a linear model there are no interaction effects between the shocks.

After time T the central bank follows a conventional interest rate rule,

$$i_t = \max\{0, \rho + \phi(\pi_t - \pi_t^*)\},$$

where $\phi > 1$ governs the response of nominal interest rates to the deviation of inflation from its target π_t^*. Different equilibrium selection rules map into different values of the inflation target at time T, π_T^*. Thus, equilibrium selection can be equivalently thought of as choosing the value of the central bank's inflation target when passive policy ends (Cochrane, 2017).

For my main analysis, I assume that $T \to \infty$. Thus, nominal interest rates do not respond to fiscal shocks of any finite duration $T_g < \infty$. In this sense I calculate the government spending multiplier under constant nominal interest rates. When I allow for $T_g \to \infty$, then I take this limit after $T \to \infty$. In appendix C I show that reversing the order of limits yields the same results.

I restrict my attention to equilibria that are bounded going forward in time. While conventional in the literature, this assumption is not innocuous (Cochrane, 2011).[3]

2.4 Discussion Specifying the government spending disturbances as a combination of a jump (at $t = 0$ and at $t = T_g$) and a diffusion ($\theta_g \geq 0$) allows me to capture many fiscal experiments in the literature. Werning (2012) and Cochrane (2017) specify the disturbance as a jump with $\theta_g = 0$ and finite $T_g < \infty$. Erceg and Lindé (2014) study a diffusion process, $\theta_g > 0$ and $T_g \to \infty$. Figure 2 plots examples of these process.

The specification for government spending does not directly nest Poisson processes in

[3]This selection criterion can still be implemented with $T \to \infty$. Suppose T is finite and there are no shocks. Then the unique forward-bounded equilibrium is the steady-state. The limit of the sequence as $T \to \infty$ is still the steady-state.

Christiano, Eichenbaum, and Rebelo (2011), Woodford (2011), Mertens and Ravn (2014), and Boneva et al. (2016). However, one can still use the framework to understand solutions for jump processes: Consider a Poisson process where government spending unexpectedly jumps to g_0 at $t = 0$. Subsequently, the process jumps to zero (an absorbing state) with intensity δ_g. At $t = 0$, the expected path for government spending is a diffusion process with $\theta_g = \delta_g$. Since the minimum state variable solution in a linear model is certainty equivalent, a Poisson process and a diffusion process with $\theta_g = \delta_g$ yield the same fiscal multiplier for $t = 0$. Hence, the minimum state variable solutions for the diffusion process can also be used to analyze solutions for Poisson processes (for $t = 0$).

3 General Solution

The model can be written as a system of linear differential equations,

$$\frac{d}{dt}\begin{pmatrix} c_t \\ \pi_t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ -\kappa^* & \rho \end{pmatrix}}_{\equiv A}\begin{pmatrix} c_t \\ \pi_t \end{pmatrix} + \begin{pmatrix} 0 \\ -\kappa^* \xi_g s_g \end{pmatrix} g_t.$$

The eigenvalues of the matrix A are

$$\lambda_{1,2} = \frac{\rho}{2} \pm \sqrt{\left(\frac{\rho}{2}\right)^2 + \kappa^*}$$

where $\lambda_1 > 0$ and $\lambda_2 < 0$ so long as prices are not perfectly rigid, $\kappa^* > 0$.

I solve the model using the elegant difference operator method introduced in Cochrane (2017). This yields a general solution for consumption,

$$c_t = \frac{\kappa^* \xi_g s_g}{\lambda_2 - \lambda_1}\left[\int_t^{T_g} e^{-\lambda_1(s-t)} g_s ds + \int_0^t e^{\lambda_2(t-s)} g_s ds\right] - \frac{1}{\lambda_2 - \lambda_1} C_1 e^{\lambda_1 t} + \frac{1}{\lambda_2 - \lambda_1} C_2 e^{\lambda_2 t}$$

subject to the unknown constants C_1 and C_2. Inflation can then be calculated from the

Euler equation,

$$\pi_t = \frac{-\kappa^* \xi_g s_g}{\lambda_2 - \lambda_1} \left[\int_t^{T_g} \lambda_1 e^{-\lambda_1 (s-t)} g_s ds + \int_0^t \lambda_2 e^{\lambda_2 (t-s)} g_s ds \right] + \frac{\lambda_1}{\lambda_2 - \lambda_1} C_1 e^{\lambda_1 t} - \frac{\lambda_2}{\lambda_2 - \lambda_1} C_2 e^{\lambda_2 t}$$

In what follows I use different selection criteria to determine the constants C_1 and C_2. I then calculate the government spending multipliers conditional on each selection criterion.

4 Government spending Multipliers

4.1 Standard new Keynesian selection criterion The conventional new Keynesian selection criterion enforces an immediate return to steady-state inflation ($\pi_t^* = 0$) when the shock disappears, $c_{T_g} = 0$ and $\pi_{T_g} = 0$. This implies,

$$C_1 = 0, \qquad C_2 = -\kappa^* \xi_g s_g \int_0^{T_g} e^{-\lambda_2 s} g_s ds$$

and yields

$$c_t = \frac{\kappa^* \xi_g s_g}{\lambda_2 - \lambda_1} \int_t^{T_g} [e^{-\lambda_1 (s-t)} - e^{-\lambda_2 (s-t)}] g_s ds > 0$$
$$\pi_t = \frac{-\kappa^* \xi_g s_g}{\lambda_2 - \lambda_1} \int_t^{T_g} [\lambda_1 e^{-\lambda_1 (s-t)} - \lambda_2 e^{-\lambda_2 (s-t)}] g_s ds > 0$$

Under the standard equilibrium selection criterion, both consumption and inflation increase with government spending. Higher government spending raises the marginal cost of production, which raises current and expected inflation. Expected real interest rates fall since nominal interest rates are constant, which induces higher consumption today rather than later. Thus, consumption expands through intertemporal substitution (Christiano et al., 2011; Cochrane, 2017).

I simplify these expressions using the process for government spending in equation (1).

The solution for consumption is,

$$c_t = \begin{cases} \frac{\kappa^* \xi_g}{\lambda_2 - \lambda_1} \left[\frac{1}{\lambda_1 + \theta_g} [1 - e^{-(\lambda_1 + \theta_g)(T_g - t)}] - \frac{1}{\lambda_2 + \theta_g} [1 - e^{-(\lambda_2 + \theta_g)(T_g - t)}] \right] s_g g_t & \text{if } 0 \leq t < T_g \\ 0 & \text{if } t \geq T_g \end{cases}$$

and the solution for inflation is

$$\pi_t = \begin{cases} \frac{-\kappa^* \xi_g}{\lambda_2 - \lambda_1} \left[\frac{\lambda_1}{\lambda_1 + \theta_g} [1 - e^{-(\lambda_1 + \theta_g)(T_g - t)}] - \frac{\lambda_2}{\lambda_2 + \theta_g} [1 - e^{-(\lambda_2 + \theta_g)(T_g - t)}] \right] s_g g_t & \text{if } 0 \leq t < T_g \\ 0 & \text{if } t \geq T_g \end{cases}$$

The government spending multiplier (for $0 \leq t < T_g$) can then be calculated from the national income accounting identity,

$$\begin{aligned} fm_t &= 1 + \frac{\frac{\partial c_t}{\partial g_t}}{s_g} \\ &= 1 + \frac{\kappa^* \xi_g}{\lambda_2 - \lambda_1} \left[\frac{1}{\lambda_1 + \theta_g + \delta_g} [1 - e^{-(\lambda_1 + \theta_g + \delta_g)(T_g - t)}] - \frac{1}{\lambda_2 + \theta_g + \delta_g} [1 - e^{-(\lambda_2 + \theta_g + \delta_g)(T_g - t)}] \right] > 1 \end{aligned}$$

Under a standard selection rule, the constant nominal interest rate government spending multiplier is always above 1, since consumption increases with government spending.

For finite T_g the government spending multiplier is also finite. It is continuous and increasing in T_g,

$$\frac{\partial fm_t}{\partial T_g} = \frac{\kappa^* \xi_g}{\lambda_2 - \lambda_1} \left[e^{-(\lambda_1 + \theta_g + \delta_g)(T_g - t)} - e^{-(\lambda_2 + \theta_g + \delta_g)(T_g - t)} \right] > 0$$

and continuous and decreasing in θ_g,

$$\begin{aligned} \frac{\partial fm_t}{\partial \theta_g} = \frac{\kappa^* \xi_g}{\lambda_2 - \lambda_1} \Bigg[&- \frac{1}{(\lambda_1 + \theta_g + \delta_g)^2} [1 - e^{-(\lambda_1 + \theta_g + \delta_g)(T_g - t)}] + \frac{1}{(\lambda_2 + \theta_g + \delta_g)^2} [1 - e^{-(\lambda_2 + \theta_g + \delta_g)(T_g - t)}] \\ &+ \frac{(T_g - t)}{\lambda_1 + \theta_g + \delta_g} e^{-(\lambda_1 + \theta_g + \delta_g)(T_g - t)} - \frac{(T_g - t)}{\lambda_2 + \theta_g + \delta_g} e^{-(\lambda_2 + \theta_g + \delta_g)(T_g - t)} \Bigg] < 0 \end{aligned}$$

Thus, for finite T_g, increasing either the duration of the government spending shock or reducing its decay rate smoothly increases the government spending multiplier. There is no discontinuity conditional on using this selection criterion.

As $T_g \to \infty$, the government spending process approaches a diffusion with decay rate θ_g.

In taking this limit, the government spending multiplier may either asymptote or explode. When the decay rate is sufficiently high, $\theta_g + \lambda_2 > 0$, then the limit of the multiplier is finite, else it is infinite,

$$\lim_{T_g \to \infty} fm_t = \begin{cases} 1 + \frac{\kappa^* \xi_g}{(\lambda_1 + \theta_g)(\lambda_2 + \theta_g)} & \text{if } \theta_g + \lambda_2 > 0 \\ \infty & \text{if } \theta_g + \lambda_2 \leq 0 \end{cases}$$

Even though the parameter space is bifurcated, there remains a sense of continuity at the $\lambda_2 + \theta_g = 0$ boundary. As $\theta_g + \lambda_2$ approaches 0 from above, then the government spending multiplier becomes unboundedly large, $\lim_{\theta_g + \lambda_2 \downarrow 0}(\lim_{T_g \to \infty} fm_t) \to \infty$. In appendix B, I formally prove that the limit is continuous by extending the definition of continuity to include $\pm\infty$.

In appendix C, I consider the case where the persistence of fiscal policy exceeds that of constant nominal interest rates, $T < T_g < \infty$. The government spending multiplier then becomes a weighted average of the government spending multiplier under constant interest rates and the government spending multiplier in normal times, with weights determined by the duration of constant nominal interest rates, T. For that case, I recover the same limits as above as $T \to \infty$. This implies that one can interchangeably take the limits, $\lim_{T_g \to \infty}(\lim_{T \to \infty} fm_t) = \lim_{T \to \infty}(\lim_{T_g \to \infty} fm_t)$, so the order of taking limits is not a source of discontinuity.

4.2 Minimum state variable criterion For Poisson processes the minimum state variable is typically invoked as a selection criterion (Christiano et al., 2011; Mertens and Ravn, 2014; Boneva et al., 2016). The minimum set of state variables is the current level of government spending g_t. Conditional on not jumping to $g_t = 0$ (the absorbing state) between t and $t + dt$, the economy starts from the same set of condition, $g_{t+dt} = g_t$. This suggests that the multiplier ought to be the same irrespective of time elapsed (again conditional on not jumping).

A similar argument can be made for a pure diffusion process ($\theta_g > 0$, $T_g \to \infty$). Between

t and $t + dt$ government spending declines by $\theta_g g_t dt$. But since the model is linear, the solution can simply be scaled with the size of g_t. This suggests the solution for a diffusion also ought to be independent of t once we condition on g_t.

The C_1 and C_2 that eliminate the time-varying terms in the solution for consumption and inflation (given $T_g \to \infty$) are,

$$C_1 = 0, \qquad\qquad C_2 = -\frac{\kappa^* \xi_g s_g}{\lambda_2 + \theta_g} g_0$$

which yields the following solutions for consumption and inflation,

$$c_t = \frac{\kappa^* \xi_g s_g}{(\lambda_1 + \theta_g)(\lambda_2 + \theta_g)} g_0$$
$$\pi_t = \frac{\theta_g \kappa^* \xi_g s_g}{(\lambda_1 + \theta_g)(\lambda_2 + \theta_g)} g_0$$

The effect of government spending on consumption and inflation may either be positive or negative depending on the sign of $\lambda_2 + \theta_g$. If $\lambda_2 + \theta_g > 0$, then consumption and inflation will increase, just like in the standard equilibrium. However, if $\lambda_2 + \theta_g < 0$, then both consumption and inflation will fall. Note that the behavior of consumption and inflation is discontinuous at the boundary. When $\lambda_2 + \theta_g$ is just above 0, consumption and inflation blow up to plus infinity. When $\lambda_2 + \theta_g = 0$ is just below zero, they blow up to minus infinity.

This behavior is mirrored in the government spending multiplier,

$$fm_t = 1 + \frac{\kappa^* \xi_g}{(\lambda_1 + \theta_g)(\lambda_2 + \theta_g)}$$

which is above 1 if $\theta_g + \lambda_2 > 0$ and below 1 if $\theta_g + \lambda_2 < 0$. As shown in figure 1, it explodes to plus infinity when $\theta_g + \lambda_2$ is just above zero and to minus infinity when $\theta_g + \lambda_2 > 0$ is just below zero.

Clearly the function $fm_t(\theta_g)$ is discontinuous at $\theta_g = -\lambda_2$ under standard definitions of continuity. As I show in appendix B, the function is also discontinuous using my extended definition of continuity.

This bifurcation result is familiar from Woodford (2011), Mertens and Ravn (2014), and

Boneva et al. (2016). The setup in these models differs from mine because their forcing process is Poisson. But because there is certainty equivalence in the linear model, the bifurcation also occurs for the expected path of government spending, which is a diffusion.

Mertens and Ravn (2014) and Boneva et al. (2016) call the $\theta_g + \lambda_2 > 0$ case the fundamental equilibrium, and the $\theta_g + \lambda_2 < 0$ case the sunspot equilibrium. The solution for the fundamental case coincides with the standard equilibrium for a diffusion process. Another way to see this is that both solutions impose the same boundary condition in the limit as $T_g \to \infty$.

However, when $\lambda_2 + \theta_g < 0$, then consumption (and the government spending multiplier) become infinitely large as $T_g \to \infty$ under the standard selection criterion. This solution is not bounded, and therefore it will not be picked by a minimum state variable criterion. Instead, the minimum state variable criterion selects a non-standard equilibrium for $\lambda_2 + \theta_g < 0$, while it selects the standard equilibrium for $\lambda_2 + \theta_g > 0$.

Thus, the bifurcation result follows from the way that equilibria are selected. Continuity in the government spending multiplier could be preserved by adopting the standard equilibrium selection for finite T_g and then taking the limit as $T_g \to \infty$. As I show next, continuity could also be preserved using Cochrane's (2017) backward-stable selection criterion throughout.

4.3 Backward-stable criterion Cochrane's (2017) backward-stable criterion looks for a solution that remains bounded as $t \to -\infty$ (as well as bounded going forward). The boundary conditions that implement the backward-stable solution are

$$C_1 = C_2 = 0$$

to eliminate the term $C_2 e^{\lambda_1 t}$, which is unstable going forward, and to eliminate the $C_2 e^{\lambda_2 t}$, which is unstable going backwards. This criterion yields solutions for consumption and inflation equal to

$$c_t = \frac{\kappa^* \xi_g s_g}{\lambda_2 - \lambda_1} \left[\int_t^{T_g} e^{-\lambda_1 (s-t)} g_s ds + \int_0^t e^{\lambda_2 (t-s)} g_s ds \right]$$
$$\pi_t = \frac{-\kappa^* \xi_g s_g}{\lambda_2 - \lambda_1} \left[\int_t^{T_g} \lambda_1 e^{-\lambda_1 (s-t)} g_s ds + \int_0^t \lambda_2 e^{\lambda_2 (t-s)} g_s ds \right]$$

Since the two integrals in the consumption equation are positive and $\lambda_2 - \lambda_1 < 0$ it follows that consumption will decline. Inflation will start out positive and become negative at some time $0 < t < T_g$. Intuitively, the decline in consumption dominates the effect of higher government spending on marginal costs. Thus, prices are expected to fall relative to $t = 0$. The expected deflation pushes up real interest rates, which validates the decline in consumption.

The backward-looking integral implies that consumption and inflation are non-zero for $t \geq T_g$,

$$c_t = \begin{cases} \frac{\kappa^* \xi_g}{\lambda_2 - \lambda_1} \left[\frac{1}{\lambda_1 + \theta_g} [1 - e^{-(\lambda_1 + \theta_g)(T_g - t)}] - \frac{1}{\lambda_2 + \theta_g} [1 - e^{(\lambda_2 + \theta_g) t}] \right] s_g g_t & \text{if } 0 \leq t < T_g \\ \frac{\kappa^* \xi_g}{\lambda_2 - \lambda_1} \left[- \frac{1}{\lambda_2 + \theta_g} [1 - e^{(\lambda_2 + \theta_g) T_g}] e^{(\lambda_2 + \theta_g)(t - T_g)} \right] s_g g_{T_g} & \text{if } t \geq T_g \end{cases}$$

$$\pi_t = \begin{cases} \frac{-\kappa^* \xi_g}{\lambda_2 - \lambda_1} \left[\frac{\lambda_1}{\lambda_1 + \theta_g} [1 - e^{-(\lambda_1 + \theta_g)(T_g - t)}] - \frac{\lambda_2}{\lambda_2 + \theta_g} [1 - e^{(\lambda_2 + \theta_g) t}] \right] s_g g_t & \text{if } 0 \leq t < T_g \\ \frac{-\kappa^* \xi_g}{\lambda_2 - \lambda_1} \left[- \frac{\lambda_2}{\lambda_2 + \theta_g} [1 - e^{(\lambda_2 + \theta_g) t}] e^{(\lambda_2 + \theta_g)(t - T_g)} \right] s_g g_{T_g} & \text{if } t \geq T_g \end{cases}$$

Nominal interest rates, however, remain at zero throughout.[4]

The government spending multiplier for $t < T_g$ is,

$$fm_t = 1 + \frac{\kappa^* \xi_g}{\lambda_2 - \lambda_1} \left[\frac{1}{\lambda_1 + \theta_g} [1 - e^{-(\lambda_1 + \theta_g)(T_g - t)}] - \frac{1}{\lambda_2 + \theta_g} [1 - e^{(\lambda_2 + \theta_g) t}] \right] < 1,$$

which is always below 1, since consumption falls when government spending increases (as shown in Cochrane (2017)).

This selection criterion also implies that small changes in the fiscal program lead to

[4]Of course, monetary policy still selects this equilibrium through the appropriate choice of π_T^* as $T \to \infty$.

continuous changes in the multiplier. First, the multiplier is smoothly decreasing in T_g,

$$\frac{\partial fm_t}{\partial T_g} = \frac{\kappa^* \xi_g}{\lambda_2 - \lambda_1}\left[e^{-(\lambda_1+\theta_g)(T_g-t)}\right] < 0$$

and changing smoothly with θ_g,

$$\begin{aligned}\frac{\partial fm_t}{\partial \theta_g} = \frac{\kappa^* \xi_g}{\lambda_2 - \lambda_1}\Bigg[&-\frac{1}{(\lambda_1+\theta_g)^2}[1-e^{-(\lambda_1+\theta_g)(T_g-t)}] + \frac{T_g - t}{\lambda_1+\theta_g}e^{-(\lambda_1+\theta_g)(T_g-t)} \\ &+\frac{1}{(\lambda_2+\theta_g)^2}[1-e^{(\lambda_2+\theta_g)t}] + \frac{t}{\lambda_2+\theta_g}e^{(\lambda_2+\theta_g)t}\Bigg].\end{aligned}$$

(This derivative may change sign, but it does so smoothly.)

The limit as $T_g \to \infty$ remains finite, irrespective of the values for $\lambda_2 + \theta_g$,

$$\lim_{T_g\to\infty} fm_t = 1 + \frac{\kappa^* \xi_g}{\lambda_2 - \lambda_1}\left[\frac{1}{\lambda_1+\theta_g} - \frac{1}{\lambda_2+\theta_g}[1-e^{(\lambda_2+\theta_g)t}]\right] < 1$$

I next let the current time t go to $+\infty$. The spirit of this exercise is to create a sense of stationarity, in that the fiscal shock has existed for a long time and is expected to continue for a long time. This is necessary to link up with the minimum state variable criterion, which is looking for a stationary solution. The limit $t \to +\infty$ yields,

$$\lim_{t\to\infty}(\lim_{T_g\to\infty} fm_t) = \begin{cases} -\infty & \text{if } \theta_g + \lambda_2 \geq 0 \\ 1 + \frac{\kappa^*\xi_g}{(\lambda_1+\theta_g)(\lambda_2+\theta_g)} & \text{if } \theta_g + \lambda_2 < 0 \end{cases}$$

When $\theta_g+\lambda_2 \geq 0$ then the multiplier now is now infinitely negative, whereas when $\theta_g+\lambda_2 < 0$ it is finite and less than 1.[5] As in the standard equilibrium, there is a sense of continuity at the boundary since $\lim_{t\to\infty}(\lim_{T_g\to\infty} fm_t)$ becomes infinitely negative as $\theta_g + \lambda_2$ approaches 0 from below. Again, I formally prove continuity in appendix B.

Further, the multiplier for $\theta_g + \lambda_2 < 0$ coincides exactly with the minimum state variable solution for the diffusion process in the sunspot case. Thus, the bifurcation for the diffusion process can be understood as a switch from the conventional equilibrium selection criterion

[5] The equilibrium explodes for $t \to +\infty$ and $\theta_g \geq -\lambda_2$ because the government spending process explodes at $t = 0$, $g_0 = \lim_{t\to\infty} e^{\theta_g t} g_t \to \infty$. The backward-stable solution does remain bounded whenever the disturbance is also bounded.

to the backward-stable equilibrium selection criterion. Rather than being a fundamental property of the model, the discontinuity arises because different equilibria are selected across the parameter space.

4.4 Numerical Example The discount rate is $\rho = 0.04$ and the slope of the Phillips curve is $\kappa^* = 0.5$. I set the elasticity of marginal cost with respect to government spending to $\xi_g = 0.2$. I pick two sets of values for t and T_g. First, I let the government spending shock to have already existed for $t = 3$ periods and to last until period $T_g = 8$. These are the thick lines in figure 3, where multipliers are plotted for the range of possible θ_g. The multipliers for both the standard and backward-stable equilibrium are finite and continuous.

I then increase t and T_g to showcase convergence towards minimum state variable solution. These are the thin lines in figure 3. As shown, increases in T_g will push the multiplier in the standard equilibria upward, and increases in T_g and t will push the multiplier in the backward-stable equilibria downward. When t and T_g become sufficiently large, then the minimum state variable solution essentially coincides with the standard equilibrium to the left of the bifurcation point and with the backward-stable equilibrium to the right of the bifurcation point.

5 Conclusion

The discontinuity in government spending multipliers plotted in figure 1 can be understood as comparing multipliers under two different selection criteria. The large positive multipliers are a limit of the standard equilibrium selection criterion, whereas the small or negative multipliers are a limit of the backward-stable selection criterion. Using either selection criterion over the entire parameter space produces continuous multipliers. Thus, the discontinuity is a consequence of equilibrium selection and not a fundamental property of the model. At a minimum, these findings suggest caution in using the minimum state variable criterion for selecting equilibria. A simple solution to avoid inconsistencies and discontinuities is to restrict the duration of shocks to be finite but arbitrarily large.

My analysis thus shows that government spending multipliers in the new Keynesian model (under a consistent equilibrium selection) are much less sensitive to the fiscal experiment than figure 1 implies. This is good news for policy analysis since it means that small changes in fiscal policy do not radically alter their efficacy.

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6 Figures

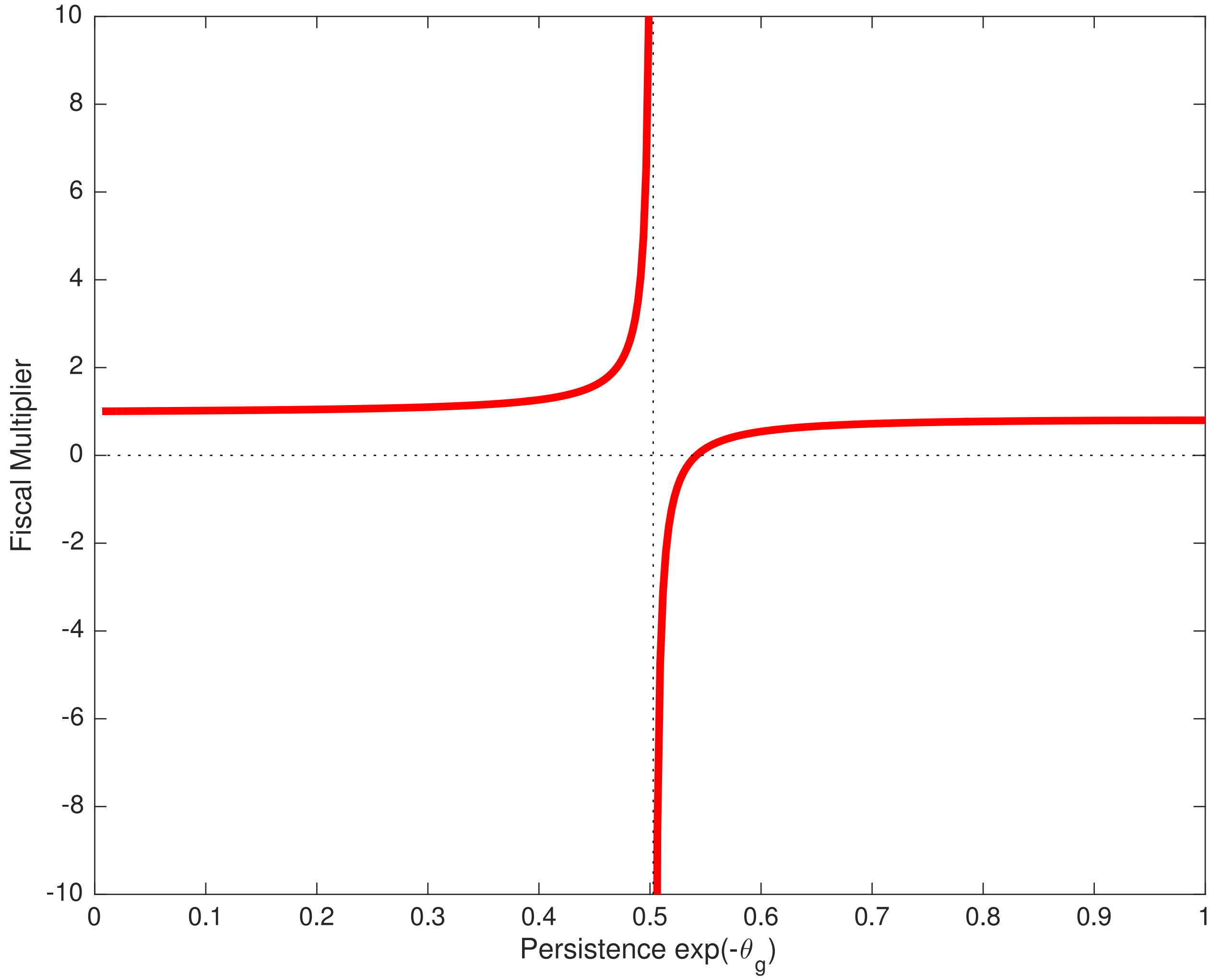

Figure 1 – Government spending multipliers under constant nominal interest rates for a government spending diffusion process with decay rate θ_g. The equilibrium is selected using the minimum state variable criterion. The parameter values are as in section 4.4.

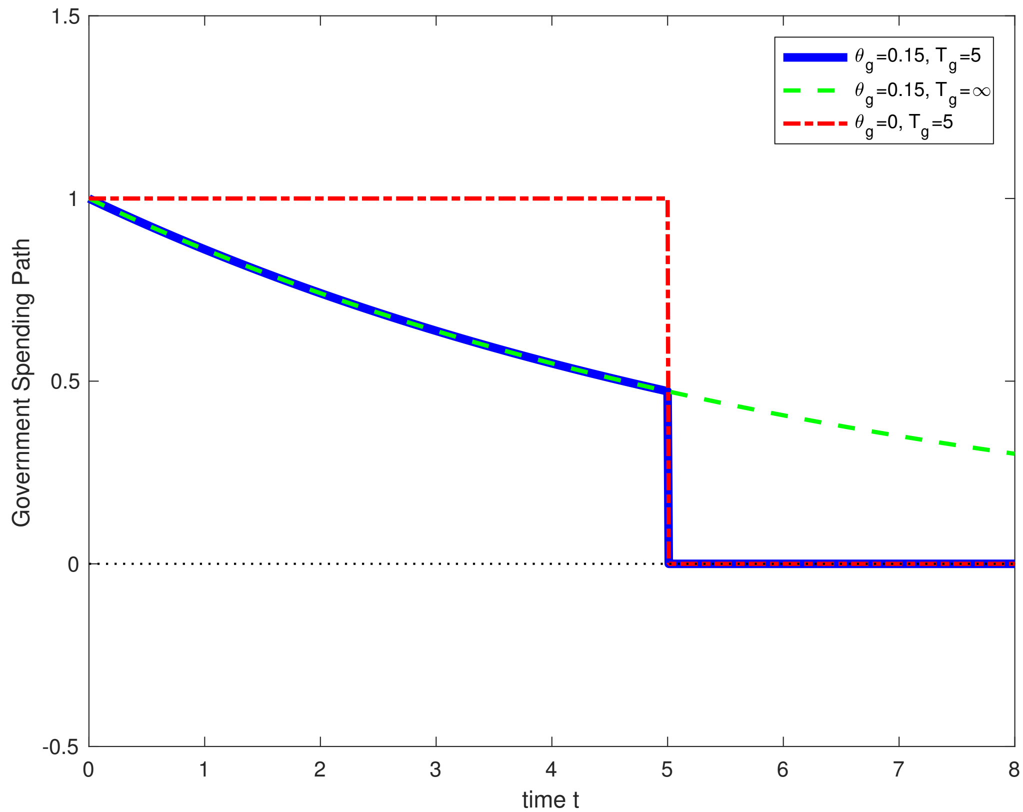

Figure 2 – Examples of government spending paths for different values of θ_g and T_g.

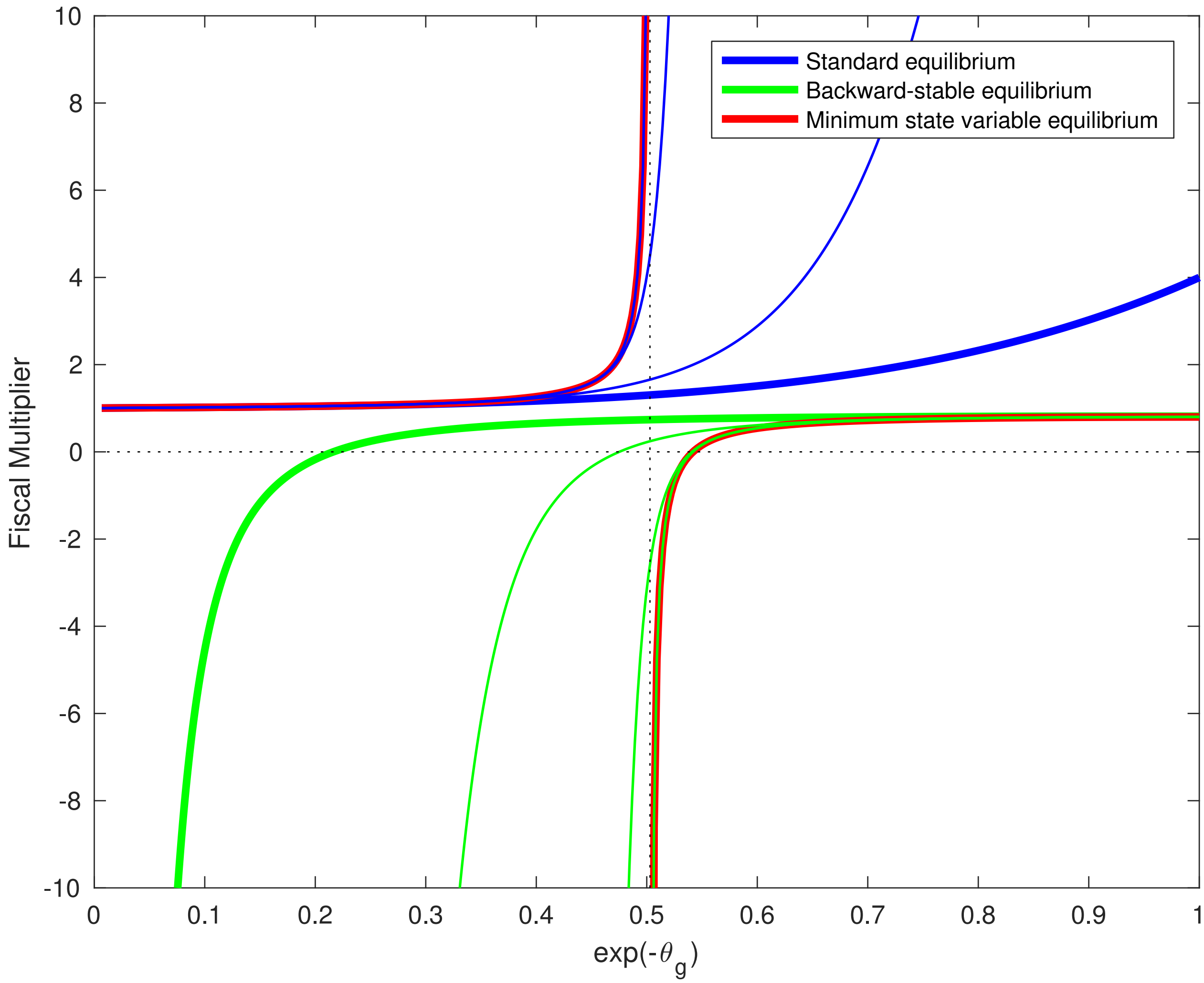

Figure 3 – Government spending multipliers under constant nominal interest rates for different equilibrium selection rules. The minimum state variable equilibrium is the same as in figure 1. It is plotted for diffusion process es with decay rate θ_g. The thick lines for the standard equilibrium and the backward-stable equilibrium are plotted for government spending processes with decay rate θ_g that have existed for $t = 3$ periods and will last until period $T_g = 8$. The thin lines for the standard and backward-stable equilibrium increase the values for t and T_g to illustrate convergence towards the minimum state variable criterion. Uses the parameter values in section 4.4.

A Model

A.1 Household Households maximize utility, which is separable preferences over consumption C_t and labor supply L_t,

$$\mathcal{U}_0 = \max \int_{t=0}^{\infty} e^{-\rho t} U(C_t, L_t) = \max \int_{t=0}^{\infty} e^{-\rho t} \left[\ln C_t - \chi \frac{L_t^{1+\psi}}{1+\psi} \right],$$

where ρ is the discount rate and ψ is the inverse Frisch elasticity. Utility is maximized subject to the period-by-period budget constraints,

$$\lambda_t : \quad \frac{d}{dt} B_t = i_t B_t + W_t L_t - P_t C_t + \Pi_t - T_t, \qquad \forall t \geq 0$$

where λ_t is the Lagrange multiplier on the budget constraint, B_t are nominal bond holdings, P_t is the nominal price of real consumption, i_t is the nominal interest rate, W_t is the common nominal wage rate across firms, Π_t are profits remitted by firms, and T_t are lump-sum taxes imposed by the government.

First order conditions for the households are as follows:

$$\begin{aligned} C_t^{-1} &= \lambda_t P_t, \\ \chi L_t^{\psi} &= \lambda_t W_t, \\ \frac{d}{dt} \lambda_t &= \rho - i_t. \end{aligned}$$

The Euler equation in the text obtains by combining the first and third equation,

$$\frac{d}{dt} C_t = (i_t - \pi_t - \rho) C_t$$

A.2 Firms Firms produce varieties indexed by i over the unit interval. Aggregate consumption is a CES aggregate over individual varieties with elasticity of substitution σ,

$$C_t = \left[\int_0^1 C_t(i)^{\frac{\sigma-1}{\sigma}} di \right]^{\frac{\sigma}{\sigma-1}},$$

which implies the aggregate price index,

$$P_t = \left[\int_0^1 C_t(i)^{1-\sigma} di \right]^{\frac{1}{1-\sigma}},$$

and relative demands,

$$C_t(i) = C_t \left[\frac{P_t(i)}{P_t} \right]^{-\sigma},$$

An analogous demand equations exist for the government. In equilibrium, output produced must equal output demanded,

$$C_t(i) + G_t(i) = Y_t(i).$$

Firms produce varieties using labor N_t,

$$Y_t(i) = N_t(i).$$

Price setting is subject to Rotemberg pricing frictions (Rotemberg, 1982). For each firm, the cost of price adjustment is $\frac{\gamma}{2}\Delta_t(i)^2 P_t Y_t$ where $\Delta_t(i)dt = d\ln P_t(i)$. An employment subsidy $\tau = \frac{1}{\sigma}$ offsets the distortions from monopolistic competition. The optimal reset prices solve the following optimization problem:

$$\max_{\{\Delta_t(i)\}_t} \int_{t=0}^{\infty} Q_{0,t}\left[P_t(i)Y_t(i) - (1-\tau)W_t Y_t(i) - \frac{\gamma}{2}\Delta_t(i)^2 P_t Y_t\right]$$
$$\text{s.t.} \quad \Delta_t(i)dt = d\ln P_t(i)$$

where $Q_{t,t+j} = e^{-\rho j}\frac{C_{t+j}^{-1}}{P_{t+j}}$ is used to evaluate future nominal cash flows.

Denote the co-state variable by $q_t(i)$. Then the first order conditions are

$$q_t(i) = \gamma\Delta_t(i)C_t^{-1}Y_t$$
$$\frac{d}{dt}q_t - \rho q_t = \left[-(1-\sigma)\left(\frac{P_t(i)}{P_t}\right)^{-\sigma} - (1-\tau)\sigma\frac{W_t}{P_t}\left(\frac{P_t(i)}{P_t}\right)^{-\sigma-1}\right]C_t^{-1}Y_t$$

Since this problem is identical for each firms, they all charge the same price $P_t(i) = P_t$. Since $\Delta_t = \pi_t$, I get,

$$q_t = \gamma\pi_t C_t^{-1}Y_t$$
$$\frac{d}{dt}q_t - \rho q_t = \left[-(1-\sigma) - (1-\tau)\sigma\frac{W_t}{P_t}\right]C_t^{-1}Y_t$$

A.3 Government The central bank sets the nominal interest rate according to an interest rate rule subject to zero lower bound constraint,

$$i_t = \max\{\rho + \phi(\pi_t - \pi_t^*), 0\}$$

where π_t^* is the inflation target.

Any subsidies to firms and any government spending is financed by lump-sum taxes within the period,

$$T_t = \tau\frac{W_t}{P_t}N_t + G_t.$$

Thus, the government runs a balanced budget each period.

Steady-state government spending is $\bar{G} = s_g\bar{Y}$.

A.4 Market clearing All markets clear if and only if

$$L_t = N_t = \int_0^1 N_t(i)di,$$
$$C_t + G_t + \frac{\gamma}{2}\pi_t^2 Y_t = Y_t,$$
$$B_t = 0.$$

A.5 Steady-state The zero inflation steady-state is:

$$\begin{aligned}
\bar{L} &= \bar{N}, \\
\bar{G}(i) + \bar{C}(i) &= \bar{Y}(i), \\
\bar{G} + \bar{C} &= \bar{Y}, \\
\bar{B} &= 0, \\
\bar{i} &= \rho, \\
\bar{\pi} &= 0, \\
M\bar{C} &= 1 \\
\frac{\bar{W}}{P} &= 1, \\
\bar{Y} &= \left(\frac{1}{\chi(1-s_g)}\right)^{\frac{1}{1+\psi}}, \\
\bar{L} &= \bar{Y}, \\
\bar{T} &= \frac{1}{\sigma}\bar{Y} + s_g\bar{Y}
\end{aligned}$$

A.6 Log-linearization The linear approximation to the Euler equation is

$$dc_t = i_t - \pi_t - \rho$$

The linear approximation of the new Keynesian Phillips curve around the zero inflation steady-state is

$$d\pi_t = \rho\pi_t - \kappa^* \left(c_t + \xi_g g_t\right)$$

where $\kappa^* \equiv \frac{\sigma-1}{\gamma}(1+\psi(1-s_g))$ and $\xi_g = \frac{\psi s_g}{(1+\psi(1-s_g))}$.

B Proofs of Continuity

I extend the real line $\mathbb{R}$ to include the points $\{+\infty, -\infty\}$, which is known as the extended real line $\bar{\mathbb{R}}$ (or extended reals). One can no longer form a metric space using the Euclidian distance since $d(x, +\infty) = +\infty \notin \mathbb{R}$ for $x \in \mathbb{R}$.

I instead treat the extended reals $\bar{\mathbb{R}}$ as a topological space and impose the order topology.[6] This topology implies that all sets of the form $[-\infty, b)$ and $(a, \infty]$ with $a, b \in \mathbb{R}$ are open in $\bar{\mathbb{R}}$. Further, these "open rays" form a subbasis for the order topology.

Under this topology, a function $g(x) : \Omega \to \bar{\mathbb{R}}$ is continuous in $x \in \Omega$, if and only if for any open ray $B = [-\infty, b)$ or $B = (a, \infty]$ with $a, b \in \mathbb{R}$, the pre-image $g^{-1}(B)$ is open in Ω.

B.1 Standard equilibrium selection criterion I need to prove that the limit of the government spending multiplier

$$g(\theta_g) = \begin{cases} 1 + \frac{\kappa^* \xi_g}{(\lambda_1+\theta_g)(\lambda_2+\theta_g)} & \text{if } \theta_g + \lambda_2 > 0 \\ \infty & \text{if } \theta_g + \lambda_2 \leq 0 \end{cases}$$

is continuous in θ_g. This function is a mapping $g(\theta_g) : \mathbb{R}_{\geq 0} \to \bar{\mathbb{R}}$.

The pre-images of the open rays $[-\infty, b)$ and $(a, \infty]$ are,

$$B = [-\infty, b) \qquad \Rightarrow g^{-1}(B) = \begin{cases} \emptyset & \text{if } b \leq 1 \\ (\frac{\rho}{2} + \sqrt{\frac{\rho^2}{4} + \kappa^* + \frac{\kappa^* \xi_g}{b-1}}, \infty) & \text{if } \infty > b > 1 \end{cases}$$

$$B = (a, \infty] \qquad \Rightarrow g^{-1}(B) = \begin{cases} \mathbb{R}_{\geq 0} & \text{if } a < 1 \\ [0, \frac{\rho}{2} + \sqrt{\frac{\rho^2}{4} + \kappa^* + \frac{\kappa^* \xi_g}{a-1}}) & \text{if } \infty > a > 1 \end{cases}$$

All pre-images $g^{-1}(B)$ are open in the non-negative reals $\mathbb{R}_{\geq 0}$, proving continuity (in this topology) of the government spending multiplier.

B.2 Minimum state variable criterion For this case, I prove that the government spending multiplier

$$g(\theta_g) \equiv 1 + \frac{\kappa^* \xi_g}{(\lambda_1 + \theta_g)(\lambda_2 + \theta_g)}$$

is discontinuous at $\theta_g = -\lambda_2 > 0$. This function is a mapping, $g(\theta_g) : \mathbb{R}_{\geq 0} \to \bar{\mathbb{R}}$.

One has to define what value the function takes at the point of discontinuity. When $g(\theta_g = -\lambda_2) = \infty$, then the preimage of the open set $(1, \infty]$ is $[-\lambda_2, \infty]$, which is not open in $R_{\geq 0}$. When $g(\theta_g = -\lambda_2) = -\infty$, then the preimage of the open set $[-\infty, 1 - \xi_g)$ is $[0, -\lambda_2]$, which is again not open in $R_{\geq 0}$. In either case, the function is not continuous in this topology.

[6] Alternatively one can use the standard δ, ε continuity proof given a suitable metric $d(x, y)$ for the space $x, y \in \bar{\mathbb{R}}$. The Eucledian distance is not a suitable metric since $d(x, \infty) = d(y, \infty) = \infty$ when $x \neq y$, but a metric space can be formed using $d(x, y) = |tan^{-1}y - tan^{-1}x|$.

B.3 Backward-stable equilibrium selection criterion I need to prove that the limit of the government spending multiplier

$$g(\theta_g) = \begin{cases} -\infty & \text{if } \theta_g \geq -\lambda_2 \\ 1 + \frac{\kappa^* \xi_g}{(\lambda_1+\theta_g)(\lambda_2+\theta_g)} & \text{if } \theta_g < -\lambda_2 \end{cases}$$

is continuous in θ_g. This function is a mapping $g(\theta_g) : \mathbb{R}_{\geq 0} \to \bar{\mathbb{R}}$.

The pre-images of the open rays $[-\infty, b)$ and $(a, \infty]$ are,

$$B = [-\infty, b) \qquad \Rightarrow g^{-1}(B) = \begin{cases} (\frac{\rho}{2} + \sqrt{\frac{\rho^2}{4} + \kappa^* + \frac{\kappa^* \xi_g}{a-1+\xi_g}}, \infty) & \text{if } b < 1 - \xi_g \\ \mathbb{R}_{\geq 0} & \text{if } \infty > b > 1 \end{cases}$$

$$B = (a, \infty] \qquad \Rightarrow g^{-1}(B) = \begin{cases} [0, \frac{\rho}{2} + \sqrt{\frac{\rho^2}{4} + \kappa^* + \frac{\kappa^* \xi_g}{b-1+\xi_g}}) & \text{if } a < 1 - \xi_g \\ \emptyset & \text{if } \infty > a > 1 \end{cases}$$

All pre-images $g^{-1}(B)$ are open in the non-negative reals $\mathbb{R}_{\geq 0}$, proving continuity (in this topology) of the government spending multiplier.

C Case $T_g > T$

C.1 Standard equilibrium selection criterion I solve the model backwards. Because there is perfect foresight and individuals know the zero lower bound will no longer bind after T, I start with normal times and then use the solution for c_T and π_T as boundary conditions for the zero lower bound.

In normal times the model is a system of linear differential equations,

$$\frac{d}{dt}\begin{pmatrix} c_t \\ \pi_t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \phi - 1 \\ -\kappa^* & \rho \end{pmatrix}}_{\equiv B} \begin{pmatrix} c_t \\ \pi_t \end{pmatrix} + \begin{pmatrix} 0 \\ -\kappa^* \xi_g s_g \end{pmatrix} g_t$$

where the eigenvalues of the matrix B are

$$\mu_{1,2} = \frac{\rho}{2} \pm \sqrt{\left(\frac{\rho}{2}\right)^2 - (\phi - 1)\kappa^*}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ so long as prices are not perfectly rigid, $\kappa^* > 0$.

Using the Cochrane (2017) difference operator method I recover the general solutions for consumption,

$$c_t = \frac{(\phi-1)\kappa^*\xi_g s_g}{\mu_2 - \mu_1}\left[-\int_t^{T_g} e^{-\mu_1(s-t)} g_s ds + \int_t^{T_g} e^{-\mu_2(s-t)} g_s ds\right] - \frac{1}{\mu_2 - \mu_1} C_1 e^{\mu_1 t} + \frac{1}{\mu_2 - \mu_1} C_2 e^{\mu_2 t}$$

The solution for inflation then follows from the Euler equation,

$$\pi_t = \frac{(\phi-1)\kappa^*\xi_g s_g}{\mu_2 - \mu_1}\left[-\mu_1\int_t^{T_g} e^{-\mu_1(s-t)} g_s ds + \mu_2\int_t^{T_g} e^{-\mu_2(s-t)} g_s ds\right] - \frac{\mu_1}{\mu_2 - \mu_1} C_1 e^{\mu_1 t} + \frac{\mu_2}{\mu_2 - \mu_1} C_2 e^{\mu_2 t}$$

C.2 Boundary conditions The conventional new Keynesian selection criterion is non-explosive behavior going forward in time. This imposes the boundary conditions $C_1 = 0$ and $C_2 = 0$.

$$\begin{aligned} c_t &= \frac{(\phi-1)\kappa^*\xi_g s_g}{\mu_2 - \mu_1}\left[-\int_t^{T_g} e^{-\mu_1(s-t)} g_s ds + \int_t^{T_g} e^{-\mu_2(s-t)} g_s ds\right] \\ \pi_t &= \frac{(\phi-1)\kappa^*\xi_g s_g}{\mu_2 - \mu_1}\left[-\mu_1\int_t^{T_g} e^{-\mu_1(s-t)} g_s ds + \mu_2\int_t^{T_g} e^{-\mu_2(s-t)} g_s ds\right] \end{aligned}$$

Substituting for the government spending process and letting $T_g \to \infty$ yields,

$$c_t = \frac{-(\phi-1)\kappa^*\xi_g s_g}{(\mu_1+\theta_g)(\mu_2+\theta_g)} g_t$$
$$\pi_t = \frac{\theta_g \kappa^*\xi_g s_g}{(\mu_1+\theta_g)(\mu_2+\theta_g)} g_t$$

The associated government spending multiplier for normal times is,

$$fm^{NT} = 1 + \frac{-(\phi-1)\kappa^*\xi_g}{(\mu_1+\theta_g)(\mu_2+\theta_g)} < 1$$

C.3 Exit conditions for zero lower bound The standard selection criterion for the zero lower bound is to enforce an immediate return to the solution for normal times upon exit. This implies the boundary conditions,

$$c_T = \frac{-(\phi-1)\kappa^*\xi_g s_g}{(\mu_1+\theta_g)(\mu_2+\theta_g)} g_T$$
$$\pi_T = \frac{\theta_g \kappa^*\xi_g s_g}{(\mu_1+\theta_g)(\mu_2+\theta_g)} g_T$$

(In the main analysis $g_T = 0$, so $\pi_T = c_T = 0$ was the boundary condition.)

Translated into the unknown coefficients C_1 and C_2, the boundary conditions are

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda_1+\theta_g} e^{-(\lambda_1+\theta_g)T} + \frac{[\lambda_2(\phi-1)-\theta_g]}{(\mu_1+\theta_g)(\mu_2+\theta_g)} e^{-(\lambda_1+\theta_g)T} \\ \frac{1}{\lambda_2+\theta_g}[e^{-(\lambda_2+\theta_g)T} - 1] + \frac{[\lambda_1(\phi-1)-\theta_g]}{(\mu_1+\theta_g)(\mu_2+\theta_g)} e^{-(\lambda_2+\theta_g)T} \end{pmatrix} \kappa^*\xi_g s_g g_0$$

The solution for consumption and inflation in the constant interest rate regime is then,

$$\begin{aligned}
c_t &= \frac{\kappa^*\xi_g s_g}{(\lambda_1+\theta_g)(\lambda_2+\theta_g)} \left[\frac{\lambda_2+\theta_g}{\lambda_2-\lambda_1}[1-e^{-\lambda_1(T-t)}] - \frac{\lambda_1+\theta_g}{\lambda_2-\lambda_1}[1-e^{-(\lambda_2+\theta_g)(T-t)}] \right] g_t \\
&+ \frac{-(\phi-1)\kappa^*\xi_g s_g}{(\mu_1+\theta_g)(\mu_2+\theta_g)} \left[\frac{[\lambda_2(\phi-1)-\theta_g]}{(\phi-1)(\lambda_2-\lambda_1)} e^{-(\lambda_1+\theta_g)(T-t)} - \frac{[\lambda_1(\phi-1)-\theta_g]}{(\phi-1)(\lambda_2-\lambda_1)} e^{-(\lambda_2+\theta_g)(T-t)} \right] g_t \\
\pi_t &= \frac{\theta_g\kappa^*\xi_g s_g}{(\lambda_1+\theta_g)(\lambda_2+\theta_g)} \left[\frac{-\lambda_1(\lambda_2+\theta_g)}{\theta_g(\lambda_2-\lambda_1)}[1-e^{-\lambda_1(T-t)}] + \frac{\lambda_2(\lambda_1+\theta_g)}{\theta_g(\lambda_2-\lambda_1)}[1-e^{-(\lambda_2+\theta_g)(T-t)}] \right] g_t \\
&+ \frac{\theta_g\kappa^*\xi_g s_g}{(\mu_1+\theta_g)(\mu_2+\theta_g)} \left[\frac{\lambda_1[\lambda_2(\phi-1)-\theta_g]}{\theta_g(\lambda_2-\lambda_1)} e^{-(\lambda_1+\theta_g)(T-t)} - \frac{\lambda_2[\lambda_1(\phi-1)-\theta_g]}{\theta_g(\lambda_2-\lambda_1)} e^{-(\lambda_2+\theta_g)(T-t)} \right] g_t
\end{aligned}$$

Expressed in this way, it is clear that the solution for consumption and inflation are weighted averages of the solution for normal times (the first line of each equation) and of the solution for permanently constant interest rates (the second line). The weights are determined by the remaining duration of passive policy, $T - t$.

The same property then also applies to the government spending multiplier,

$$fm_t = 1 + \frac{\kappa^* \xi_g}{(\lambda_1+\theta_g)(\lambda_2+\theta_g)} \left[\frac{\lambda_2+\theta_g}{\lambda_2-\lambda_1}[1-e^{-\lambda_1(T-t)}] - \frac{\lambda_1+\theta_g}{\lambda_2-\lambda_1}[1-e^{-(\lambda_2+\theta_g)(T-t)}] \right]$$
$$+ \frac{-(\phi-1)\kappa^*\xi_g}{(\mu_1+\theta_g)(\mu_2+\theta_g)} \left[\frac{[\lambda_2(\phi-1)-\theta_g]}{(\phi-1)(\lambda_2-\lambda_1)} e^{-(\lambda_1+\theta_g)(T-t)} - \frac{[\lambda_1(\phi-1)-\theta_g]}{(\phi-1)(\lambda_2-\lambda_1)} e^{-(\lambda_2+\theta_g)(T-t)} \right]$$

Whether this government spending multiplier is above 1 or below 1 depends on T. It is below 1 for $t = T$ and above 1 for $T \to \infty$. By continuity, there exists a $t < \tilde{T} < \infty$ such that the government spending multiplier is exactly 1.

Finally, I can recover the limits for the diffusion process under a permanent liquidity trap when I let $T \to \infty$,

$$\lim_{T\to\infty} fm_t = \begin{cases} 1 + \frac{\kappa^*\xi_g}{(\lambda_1+\theta_g)(\lambda_2+\theta_g)} & \text{if } \theta_g > -\lambda_2 \\ \infty & \text{if } \theta_g \le -\lambda_2 \end{cases}$$

This limit is the same as the limit in section 4.1, which I obtained by first taking $T \to \infty$ and then $T_g \to \infty$. Thus, the order of limits is irrelevant for the outcome.